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Conservation laws of semidiscrete canonical Hamiltonian equations

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Abstract

There are many evolution partial differential equations which can be cast into Hamiltonian form. Conservation laws of these equations are related to one-parameter Hamiltonian symmetries admitted by the PDEs. The same result holds for semidiscrete Hamiltonian equations. In this paper we consider semidiscrete canonical Hamiltonian equations. Using symmetries, we find conservation laws for the semidiscretized nonlinear wave equation and Schrödinger equation.

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1. Introduction

Many partial differential equations (PDEs) of non-dissipative continuum mechanics can be presented in Hamiltonian form (see [1] and references therein). It is well known that conservation laws of Hamiltonian PDEs are related to one-parameter Hamiltonian symmetries [2]. The analogue of this result holds in the semidiscrete case with no change in the statement although the framework must be modified [3].

In the present paper we examine canonical Hamiltonian equations

$$\boldsymbol{v}_t = \frac{\delta \mathcal{H}}{\delta \boldsymbol{w}} \qquad \boldsymbol{w}_t = -\frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} \tag{1.1}$$

which form a special type of Hamiltonian equations. For instance, the nonlinear wave equation

$$v_{tt} = \Delta v - V'(v) \tag{1.2}$$

and the nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \Delta\psi + F'(|\psi|^2)\psi = 0 \tag{1.3}$$

can be rewritten in the form (1.1). We will introduce semidiscrete analogues of the canonical Hamiltonian PDEs and show how one can use the Hamiltonian form of Noether's theorem to find conservation laws for these equations.

To find conservation laws of semidiscrete equations with the help of Noether's theorem, we need to know symmetries in the evolutionary form. It is possible to find semidiscrete equations which admit a known symmetry group. One can use the method of finite-difference invariants [4], i.e. to approximate the considered differential equation by differential–difference invariants of the admitted symmetry group. The method was proposed for difference equations, but it can be easily adopted for semidiscrete equations. Alternatively, it may be possible to find a discrete realization of the considered symmetry algebra. This realization can be used to construct invariant differential–difference equations [5]. However, it is not guaranteed that the obtained invariant semidiscrete equations can be cast into Hamiltonian form. For these reason we will semidiscretize the equations, preserving the Hamiltonian form, and then look for admitted symmetries.

Different methods which can be used to find symmetries of discrete and semidiscrete equations are discussed, for example, in [6]. The most successful application was shown for linear difference equations [6–8], where for the considered equations symmetry algebras isomorphic to those of the underlying continuous equations were found. Nevertheless, it is not known how to find all symmetries admitted by nonlinear semidiscrete equations.

One of the possibilities, which we exploit in this paper, is to make use of the admitted Lie point symmetries. Many Lie point symmetries of semidiscrete equations can be easily found as Lie point symmetries of the underlying continuous equations preserved under the space discretization. In both continuous and semidiscrete cases these symmetries are given by the same vector fields. Using factorization, one can obtain corresponding evolutionary operators. If the latter are Hamiltonian symmetries, they let us find conservation laws. Obviously, for our purpose we will not be interested in all symmetries but only in Hamiltonian ones.

The layout of the paper is as follows. In section 2 we briefly introduce Hamiltonian equations and specify canonical Hamiltonian equations. Symmetries and the Hamiltonian form of Noether's theorem are discussed in section 3. In sections 4 and 5 we examine semidiscretizations of equations (1.2) and (1.3) and find their conservation laws. In final section 6 we make concluding remarks. In particular, we mention the connection between Euler–Lagrange equations and canonical Hamiltonian equations.

2. Hamiltonian equations

For simplicity we will consider the case of one space coordinate x. We assume that the solutions are sufficiently smooth that all variational derivatives tend to zero as the solution tends to zero and the solution and a number of its space derivatives tend to zero as $|x| \to \infty$. We suppose that the solution decays fast enough so that all integrals and sums make sense.

2.1. Hamiltonian partial differential equations

Many systems of evolution equations

$$\boldsymbol{u}_t = K(x, \boldsymbol{u}^{(m)})$$

where $\boldsymbol{u}$ denotes N dependent variables $\boldsymbol{u} = (u^1, u^2, \dots, u^N)^T$ and $\boldsymbol{u}^{(m)} = (\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m)$ represents $\boldsymbol{u}$ and a finite set of derivatives of $\boldsymbol{u}$ with respect to space coordinate x, can be cast into the Hamiltonian form

$$\boldsymbol{u}_t = \mathcal{D}\left(\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\right) \qquad \mathcal{H}[\boldsymbol{u}] = \int H(x, \boldsymbol{u}^{(n)})\,\mathrm{d}x \qquad \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} = \left(\frac{\delta\mathcal{H}}{\delta u^1}, \frac{\delta\mathcal{H}}{\delta u^2}, \dots, \frac{\delta\mathcal{H}}{\delta u^N}\right)^T \quad (2.1)$$

with the help of the Hamiltonian functional $\mathcal{H}[\boldsymbol{u}]$, variational operator $\delta \cdot /\delta \boldsymbol{u}$ and the linear operator $\mathcal{D}$ [2]. We denote as $\mathcal{F}$ the space of functionals

$$\int P(t, x, \boldsymbol{u}^{(k)})\, \mathrm{d}x \qquad k \in \mathbb{N}.$$

The operator $\mathcal{D}$ must be Hamiltonian, i.e. it forms the Poisson bracket

$$\{\mathcal{P}, \mathcal{L}\} = \int \left(\frac{\delta \mathcal{P}}{\delta \boldsymbol{u}}\right)^T \mathcal{D} \left(\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}}\right) \mathrm{d}x \tag{2.2}$$

satisfying the conditions of *skew-symmetry*

$$\{\mathcal{P}, \mathcal{L}\} = -\{\mathcal{L}, \mathcal{P}\} \tag{2.3}$$

and the *Jacobi identity*

$$\{\{\mathcal{P}, \mathcal{L}\}, \mathcal{R}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{L}\} + \{\{\mathcal{L}, \mathcal{R}\}, \mathcal{P}\} = 0 \tag{2.4}$$

for all functionals $\mathcal{P}, \mathcal{L}, \mathcal{R} \in \mathcal{F}$.

The variational derivatives of a functional can be found by the action of the Euler operators on the integrand

$$\begin{aligned} \frac{\delta \mathcal{H}}{\delta u^i} &= E^i(H) \\ E^i &= \frac{\partial \cdot}{\partial u^i} - D_x \left(\frac{\partial \cdot}{\partial u^i_1}\right) + D_x^2 \left(\frac{\partial \cdot}{\partial u^i_2}\right) + \cdots + (-1)^n D_x^n \left(\frac{\partial \cdot}{\partial u^i_n}\right) + \cdots \end{aligned} \tag{2.5}$$

where D_x is the total space derivative operator.

2.2. Canonical Hamiltonian equations

Canonical Hamiltonian equations form a subset of equations (2.1) characterized by an even-dimensional space of dependent variables $N = 2n$, $\boldsymbol{u} = (v^1, \ldots, v^n, w^1, \ldots, w^n)^T$, and the canonical Hamiltonian operator

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \tag{2.6}$$

where I_n is the $n \times n$ identity matrix and 0_n is the $n \times n$ zero matrix. Thus these Hamiltonian equations have the form (1.1). It is easy to see that the Poisson bracket generated by operator J

$$\{\mathcal{P}, \mathcal{L}\} = \int \sum_{i=1}^{n} \left(\frac{\delta \mathcal{P}}{\delta v^i}\frac{\delta \mathcal{L}}{\delta w^i} - \frac{\delta \mathcal{P}}{\delta w^i}\frac{\delta \mathcal{L}}{\delta v^i}\right) \mathrm{d}x$$

satisfies skew-symmetry (2.3) and Jacobi identity (2.4).

2.3. Semidiscrete Hamiltonian equations

Given a Hamiltonian PDE, we attempt to discretize both the Poisson bracket and the Hamiltonian functional so that we preserve Hamiltonian structure.

To consider semidiscrete equations we introduce a two-dimensional mesh which is uniform in space and continuous in time. Let us denote the mesh points as $\{x_i(t)\}$, $i \in \mathbb{Z}$, $t \geqslant 0$ and define mesh Ω by two conditions:

$$\Omega: \quad x_{i+1}(t) - x_i(t) = x_i(t) - x_{i-1}(t) \qquad x_i(t+\tau) = x_i(t) \qquad i \in \mathbb{Z} \quad t, \tau \geqslant 0. \tag{2.7}$$

The first equation requires the space mesh to be uniform for any fixed time. The second equation requires that only vertical mesh lines in the time–space plane are considered.

Now we can introduce discrete space derivatives $\underset{h}{u}{}^i_1 = \underset{+h}{D} u^i$, $\underset{h}{u}{}^i_2 = \underset{-h}{D}\underset{h}{u}{}^i_1$, ..., $\underset{h}{u}{}^i_{2k+1} = \underset{+h}{D}\underset{h}{u}{}^i_{2k}$, $\underset{h}{u}{}^i_{2k+2} = \underset{-h}{D}\underset{h}{u}{}^i_{2k+1}$, ..., $i = 1, \dots, N$, where $\underset{+h}{D}$ and $\underset{-h}{D}$ are the right and left discrete differentiation operators

$$\underset{+h}{D} = \frac{S_+ - 1}{h} \qquad \underset{-h}{D} = \frac{1 - S_-}{h} \tag{2.8}$$

defined with the help of the right shift S_+ and left shift S_- operators,

$$S_+ f(x) = f(x+h) \qquad S_- f(x) = f(x-h). \tag{2.9}$$

We will consider the space of discrete derivatives $\underset{h}{\boldsymbol{u}}{}^{(m)} = (\boldsymbol{u}, \underset{h}{\boldsymbol{u}}{}_1, \underset{h}{\boldsymbol{u}}{}_2, \dots, \underset{h}{\boldsymbol{u}}{}_m)$ and functionals of the form

$$\underset{h}{\mathcal{P}} = \sum_{\Omega} \underset{h}{P}\,(t, x, h, \underset{h}{\boldsymbol{u}}{}^{(m)})h \tag{2.10}$$

where the summation is taken over all space points of mesh Ω for some fixed time. We denote the space of such functionals as $\underset{h}{\mathcal{F}}$.

We assume that the Hamiltonian operator $\mathcal{D}$ can be approximated by an operator $\underset{h}{\mathcal{D}}$ such that the discrete bracket

$$\{\underset{h}{\mathcal{P}}, \underset{h}{\mathcal{L}}\}_h = \sum_{\Omega} \left(\frac{\delta \underset{h}{\mathcal{P}}}{\delta \boldsymbol{u}}\right)^T \underset{h}{\mathcal{D}} \left(\frac{\delta \underset{h}{\mathcal{L}}}{\delta \boldsymbol{u}}\right) h \tag{2.11}$$

defines a Poisson bracket for functionals from $\underset{h}{\mathcal{F}}$, i.e. the bracket $\{\cdot, \cdot\}_h$ is skew-symmetric and satisfies the Jacobi identity. Then we can choose some approximation $\underset{h}{\mathcal{H}}$ of the Hamiltonian $\mathcal{H}$ to obtain the set of semidiscrete Hamiltonian equations

$$\dot{\boldsymbol{u}}_j = \underset{h}{\mathcal{D}} \left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta \boldsymbol{u}_j}\right) \qquad j \in \mathbb{Z} \tag{2.12}$$

which approximate equation (2.1) on mesh (2.7).

For the space discretization of the canonical Hamiltonian PDEs we can keep the canonical operator J since it generates a discrete Poisson bracket, namely

$$\{\underset{h}{\mathcal{P}}, \underset{h}{\mathcal{L}}\}_h = \sum_{\Omega} \sum_{i=1}^{n} \left(\frac{\delta \underset{h}{\mathcal{P}}}{\delta v^i} \frac{\delta \underset{h}{\mathcal{L}}}{\delta w^i} - \frac{\delta \underset{h}{\mathcal{P}}}{\delta w^i} \frac{\delta \underset{h}{\mathcal{L}}}{\delta v^i} \right) h$$

and take a discretization $\underset{h}{\mathcal{H}}$ of the Hamiltonian functional $\mathcal{H}$. This procedure provides us with the semidiscrete canonical Hamiltonian equations,

$$\dot{\boldsymbol{v}}_j = \frac{\delta \underset{h}{\mathcal{H}}}{\delta \boldsymbol{w}_j} \qquad \dot{\boldsymbol{w}}_j = -\frac{\delta \underset{h}{\mathcal{H}}}{\delta \boldsymbol{v}_j} \qquad j \in \mathbb{Z} \tag{2.13}$$

where we have used vector notation $\boldsymbol{v} = (v^1, \dots, v^n)^T$, $\boldsymbol{w} = (w^1, \dots, w^n)^T$.

3. Symmetries and conservation laws

3.1. Invariance of semidiscrete equations

Let $\underset{h}{Z}$ be the space of sequences of variables $(t, x, h, \boldsymbol{u}, \underset{h}{\boldsymbol{u}}_1, \underset{h}{\boldsymbol{u}}_2, \ldots)$ and $\underset{h}{\mathcal{A}}$ be the space of analytic functions of a finite number of variables z from $\underset{h}{Z}$.

Invariance of the semidiscrete equations

$$\dot{\boldsymbol{u}} = \boldsymbol{F}(z) \qquad F_i \in \underset{h}{\mathcal{A}} \tag{3.1}$$

defined in the points of some two-dimensional mesh Ω was considered in [3]. Symmetries of equations (3.1) are transformations generated by vector fields of the form

$$X = \xi^t(z)\frac{\partial}{\partial t} + \xi^x(z)\frac{\partial}{\partial x} + \eta^i(z)\frac{\partial}{\partial u^i} + \cdots \qquad \xi^t, \xi^x, \eta^i \in \underset{h}{\mathcal{A}} \tag{3.2}$$

which leave the equations and the mesh invariant. The infinitesimal criterion of invariance can be presented by the three conditions

$$X(\dot{\boldsymbol{u}} - \boldsymbol{F}(z)) = 0 \tag{3.3}$$

$$\underset{-h}{D}\,\underset{+h}{D}(\xi^x) = 0 \qquad D_t(\xi^x) = 0 \tag{3.4}$$

which are to be satisfied on the solutions of (3.1). Condition (3.3) requires the invariance of equations (3.1), while conditions (3.4) require that of mesh Ω. The operator X must be prolonged on all variables appearing in equation (3.1)

$$X = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \eta^i\frac{\partial}{\partial u^i} + \phi^i\frac{\partial}{\partial \dot{u}^i} + \underset{h}{\zeta}{}^i_1\frac{\partial}{\partial \underset{h}{u}{}^i_1} + \underset{h}{\zeta}{}^i_2\frac{\partial}{\partial \underset{h}{u}{}^i_2} + \cdots + \underset{+h}{D}(\xi^x)\frac{\partial}{\partial h}. \tag{3.5}$$

On the uniform in space grid the coefficients of the prolonged operator are defined by the prolongation formulae

$$\begin{aligned} &\phi^i = D_t(\eta^i) - \dot{u}^i D_t(\xi^t) - u^i_1 D_t(\xi^x) \qquad \underset{h}{\zeta}{}^i_1 = \underset{+h}{D}(\eta^i) - S_+(\dot{u}^i)\underset{+h}{D}(\xi^t) - \underset{h}{u}{}^i_1\underset{+h}{D}(\xi^x) \\ &\underset{h}{\zeta}{}^i_2 = \underset{-h}{D}\,\underset{+h}{D}(\eta^i) - 2\,\underset{h}{u}{}^i_2\underset{+h}{D}(\xi^x) - \frac{1}{h}S_+(\dot{u}^i)\underset{+h}{D}(\xi^t) + \frac{1}{h}S_-(\dot{u}^i)\underset{-h}{D}(\xi^t) \qquad \cdots. \end{aligned} \tag{3.6}$$

Note that $u^i_1 = D_x(u^i)$ is the 'continuous' derivative. It is assumed to be in some discrete representation, for example, $\underset{h}{\tilde{D}}{}_0(u)$, which will be introduced below.

Let us note that prolongation formulae for discrete derivatives are obtained with the help of series expansions of these derivatives. For example, the coefficient $\underset{h}{\zeta}{}^i_1$ is found using the expansion

$$\underset{h}{u}{}^{i*}_1 = \frac{u^{i*}(t, x+h) - u^{i*}(t, x)}{h^*} = u^{i*}_1 + \frac{h^*}{2!}u^{i*}_2 + \cdots$$

where $h^* = x^*(t, x+h) - x^*(t, x)$ and the operation $*$ denotes the infinitesimal Lie transformation of the considered variables

$$t^* = t + a\xi^t + \cdots \qquad x^* = x + a\xi^x + \cdots \qquad u^{i*} = u^i + a\eta^i + \cdots$$

corresponding to the transformation group parameter a. We refer an interested reader to [9, 10] for details.

Operators of the form (3.2) are called Lie–Bäcklund (or *generalized*) symmetries. It is a difficult task to find Lie–Bäcklund operators admitted by discrete equations. However, Lie point symmetries

$$X = \xi^t(t,x,\boldsymbol{u})\frac{\partial}{\partial t} + \xi^x(t,x,\boldsymbol{u})\frac{\partial}{\partial x} + \eta^i(t,x,\boldsymbol{u})\frac{\partial}{\partial u^i} + \cdots \tag{3.7}$$

where coefficients ξ^t, ξ^x and η depend only on dependent and independent variables, are easier to detect since such symmetries are given by the same vector fields in the continuous and discrete cases [9]. Practically, one can check whether Lie point symmetries admitted by the underlying PDEs are admitted by the semidiscrete equations or not. Although this procedure does not guarantee that we find all symmetries, it lets us avoid solving discrete determining equations.

3.2. *Factorization of operators*

Following [9, 10], let us consider a special operation of left multiplication of a Lie–Bäcklund operator by an analytic function $\tilde{\xi}(z) \in \underset{h}{\mathcal{A}}$:

$$\tilde{\xi} * X = \tilde{\xi}\xi^t\frac{\partial}{\partial t} + \tilde{\xi}\xi^x\frac{\partial}{\partial x} + \tilde{\xi}\eta^i\frac{\partial}{\partial u^i} + \cdots + \underset{+h}{D}(\tilde{\xi}\xi^x)\frac{\partial}{\partial h}. \tag{3.8}$$

The first coordinates in operator $\tilde{\xi} * X$ are multiplied by $\tilde{\xi}(z)$, while the remaining coordinates are computed according to the prolongation formulae (3.6).

The operator

$$\xi^t(z) * D_t + \xi^x(z) * \underset{h}{D}_x \tag{3.9}$$

where D_t is the total time derivative operator and $\underset{h}{D}_x$ is a discrete presentation of the total space derivative operator D_x, plays the role of an ideal of the Lie algebra of operators (3.2) [3]. There are several possibilities to choose operator $\underset{h}{D}_x$. One can take a representation based on the right or left discrete derivative [9, 10]

$$\begin{aligned} D^+ &= \frac{\partial}{\partial x} + \underset{+h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \cdots \qquad & \underset{+h}{\tilde{D}} &= \sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{D}^n \\ D^- &= \frac{\partial}{\partial x} + \underset{-h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \cdots \qquad & \underset{-h}{\tilde{D}} &= \sum_{n=1}^{\infty}\frac{h^{n-1}}{n}\underset{-h}{D}^n \end{aligned} \tag{3.10}$$

or the discrete representation based on the central-difference derivative [3]

$$D^0 = \frac{\partial}{\partial x} + \underset{h}{\tilde{D}}_0(u)\frac{\partial}{\partial u} + \cdots \qquad \underset{h}{\tilde{D}}_0 = \sum_{k=0}^{\infty}\alpha_{2k+1}h^{2k}\underset{h}{D}_0^{2k+1} \qquad \underset{h}{D}_0 = \frac{S_+ - S_-}{2h} \tag{3.11}$$

with coefficients

$$\alpha_{2k+1} = (-1)^k\frac{1}{2}\frac{3}{4}\cdots\frac{2k-1}{2k}\frac{1}{2k+1} = \frac{(-1)^k}{2k+1}\frac{(2k-1)!!}{2^k k!}$$

$$(2k-1)!! = 1\cdot 3\cdot 5\cdots(2k-1).$$

Let us mention that the operator $\underset{h}{\tilde{D}}_0$ can be presented in terms of powers of the shift operator S_+:

$$\underset{h}{\tilde{D}}_0 = \sum_{k=-\infty}^{\infty}c_k S_k \qquad S_k = (S_+)^k \tag{3.12}$$

with convergent coefficients c_k. To find this representation we rewrite this operator as

$$\underset{h}{\tilde{D}}{}_0 = \frac{1}{2h}\sum_{k=0}^{\infty}\frac{\alpha_{2k+1}}{2^{2k}}(S_+ - S_-)^{2k+1}. \tag{3.13}$$

Using

$$\begin{aligned}(S_+ - S_-)^{2k+1} &= \sum_{j=0}^{k}(-1)^j\frac{(2k+1)!}{j!(2k+1-j)!}S_{2k+1-2j} + \sum_{j=k+1}^{2k+1}(-1)^j\frac{(2k+1)!}{j!(2k+1-j)!}S_{2k+1-2j}\\ &= \sum_{p=0}^{k}(-1)^{k-p}\frac{(2k+1)!}{(k-p)!(k+1+p)!}S_{2p+1}\\ &\quad+\sum_{p=0}^{k}(-1)^{k+1+p}\frac{(2k+1)!}{(k-p)!(k+1+p)!}S_{-2p-1}\\ &= (-1)^k\sum_{p=0}^{k}(-1)^p\frac{(2k+1)!}{(k-p)!(k+1+p)!}(S_{2p+1} - S_{-2p-1})\end{aligned}$$

we obtain

$$\underset{h}{\tilde{D}}{}_0 = \frac{1}{2h}\sum_{p=0}^{\infty}c_p(S_{2p+1} - S_{-2p-1}) \tag{3.14}$$

where the coefficients are given by the series

$$\begin{aligned}c_p &= (-1)^p\sum_{k=p}^{\infty}(-1)^k\frac{\alpha_{2k+1}}{2^{2k}}\frac{(2k+1)!}{(k-p)!(k+1+p)!}\\ &= (-1)^p\sum_{k=p}^{\infty}\frac{1}{k+1+p}\frac{1}{2^{2k}}\frac{((2k-1)!!)^2}{(k-p)!(k+p)!}.\end{aligned} \tag{3.15}$$

Lemma 3.1. *The series (3.15) defining coefficients c_p for $p \in \mathbb{N}$ are convergent.*

Proof. Using $(k+p)!(k-p)! \geqslant (k!)^2$ we obtain

$$|c_p| \leqslant \sum_{k=p}^{\infty}\frac{1}{k+1+p}\left(\frac{(2k)!}{(2^k(k)!)^2}\right)^2.$$

We use the bounds on the factorial provided by Stirling's expansion

$$n^n\exp(-n)\sqrt{2\pi n} < n! < n^n\sqrt{2\pi n}\exp\left(-n+\frac{1}{12n}\right) \qquad \text{for} \quad n \in \mathbb{N}$$

to obtain the inequality

$$\frac{1}{k+1+p}\left(\frac{(2k)!}{(2^k(k)!)^2}\right)^2 < \frac{1}{\pi k(k+1+p)}\exp\left(\frac{1}{12k}\right) \qquad \text{for} \quad k \in \mathbb{N}$$

that ensures that the series defining c_p converges, moreover $c_p \sim 1/|p|$ as $p \to \infty$. □

One can easily check that the operators $\underset{+h}{\tilde{D}}$ and $\underset{-h}{\tilde{D}}$ cannot be presented in the form (3.12) with convergent coefficients.

Lemma 3.2. *The operator* $\tilde{\underset{h}{D}}_0$ *is skew-adjoint, i.e.*

$$\sum_{i=-\infty}^{\infty} v_i \tilde{\underset{h}{D}}_0 u_i h = - \sum_{i=-\infty}^{\infty} u_i \tilde{\underset{h}{D}}_0 v_i h$$

for $\{u_i\}$, $\{v_i\}$, $i \in \mathbb{Z}$ *such that* $u_i, v_i \to 0$ *as* $i \to \infty$.

Proof. The equality can be checked with the help of (3.14). □

In what follows we will use D^0. With the help of the ideal $\xi^t * D_t + \xi^x * D^0$ we can find the evolutionary operator corresponding to the operator (3.2),

$$\bar{X} = X - \xi^t * D_t - \xi^x * D^0 = \bar{\eta}^i \frac{\partial}{\partial u^i} + \cdots \qquad \bar{\eta}^i = \eta^i - \xi^t \dot{u}^i - \xi^x \tilde{\underset{h}{D}}_0(u^i). \tag{3.16}$$

On the solutions of equations (2.12) we can exclude time derivatives $\dot{u}^i$, $i = 1, \ldots, N$ from the coefficients of the evolutionary operator.

It is important to note that although the factorization let us find evolutionary operators corresponding to given Lie point operators, generally, the obtained evolutionary operators do not have to be admitted by the semidiscrete equations which admit the original non-evolutionary operators. In the continuous case $\xi^t * D_t = \xi^t \cdot D_t$ and $\xi^x * D_x = \xi^x \cdot D_x$, i.e. the operation $*$ is equivalent to a left multiplication of the prolonged operator [9]. The operators D_t and D_x are admitted by all differential equations and, consequently, the operators $\xi^t * D_t$ and $\xi^x * D_x$ are also admitted by all equations. It follows that an evolutionary operator is admitted if it corresponds to an admitted non-evolutionary operator [2].

This result does not hold in the semidiscrete case. In the general case operator $\xi^t * D_t = \xi^t \cdot D_t$ is admitted, but the operator $\xi^x * D^0$ is not. Thus, having obtained an evolutionary operator, one has to check that this operator or the operator $\xi^x * D^0$, which is used in the factorization, is admitted by the considered semidiscrete equations.

We can provide only a very restricted class of operators $\xi^x * D^0$ which are admitted by an arbitrary discrete equation.

Lemma 3.3. *For* ξ^x *such that* $\underset{+h}{D}(\xi^x) = 0$ *we have*

$$\xi^x * D^0 = \xi^x \cdot D^0$$

and, consequently, this operator is admitted by all difference equations.

Proof. The result follows from the prolongation formulae. □

Remark. The condition $\underset{+h}{D}(\xi^x) = 0$ is very restrictive, but its multi-dimensional analogue

$$\underset{+h_i}{D}(\xi^{x_i}) = 0$$

where x_i is an independent space variable and $\underset{+h_i}{D}$ is the right discrete derivative with respect to x_i, leaves more freedom. For example, it allows rotations in $X_i X_j$, $i \neq j$ planes

$$X_{i,j} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}.$$

3.3. Conservation laws

For a system of Hamiltonian equations (2.13) considered on the grid (2.7) we have the following types of the conservation laws:

3.3.1. Conservation of symplecticity. Due to their canonical form semidiscrete equations possess the conservation of symplecticity

$$\frac{\mathrm{d}}{\mathrm{d}t}\underset{h}{\boldsymbol{\omega}} = 0 \qquad \underset{h}{\boldsymbol{\omega}} = \sum_{j=-\infty}^{\infty} \mathrm{d}\boldsymbol{v}_j \wedge \mathrm{d}\boldsymbol{w}_j\, h = \sum_{j=-\infty}^{\infty}\sum_{i=1}^{n} \mathrm{d}v_j^i \wedge \mathrm{d}w_j^i\, h \tag{3.17}$$

where $\mathrm{d}\boldsymbol{v}_j = (\mathrm{d}v_j^1, \ldots, \mathrm{d}v_j^n)^T$ and $\mathrm{d}\boldsymbol{w}_j = (\mathrm{d}w_j^1, \ldots, \mathrm{d}w_j^n)^T$ are solutions of the variational equations

$$\mathrm{d}\dot{v}_j^i = \sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial w_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_j^i}\right)\mathrm{d}w_{j+l}^k + \sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial v_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_j^i}\right)\mathrm{d}v_{j+l}^k$$

$$\mathrm{d}\dot{w}_j^i = -\sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial w_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_j^i}\right)\mathrm{d}w_{j+l}^k - \sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial v_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_j^i}\right)\mathrm{d}v_{j+l}^k.$$

We suppose that $\mathrm{d}\boldsymbol{v}_j,\ \mathrm{d}\boldsymbol{w}_j \to 0$ as $j \to \infty$. Differentiating the 2-form $\underset{h}{\boldsymbol{\omega}}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\underset{h}{\boldsymbol{\omega}} &= \sum_{j=-\infty}^{\infty}\sum_{i=1}^{n} \mathrm{d}\dot{v}_j^i \wedge \mathrm{d}w_j^i\, h + \sum_{j=-\infty}^{\infty}\sum_{i=1}^{n} \mathrm{d}v_j^i \wedge \mathrm{d}\dot{w}_j^i\, h \\
&= \sum_{j,l=-\infty}^{\infty}\sum_{i,k=1}^{n} \frac{\partial}{\partial w_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_j^i}\right)\mathrm{d}w_{j+l}^k \wedge \mathrm{d}w_j^i\, h \\
&\quad + \sum_{j,l=-\infty}^{\infty}\sum_{i,k=1}^{n} \frac{\partial}{\partial v_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_j^i}\right)\mathrm{d}v_{j+l}^k \wedge \mathrm{d}w_j^i\, h \\
&\quad - \sum_{j,l=-\infty}^{\infty}\sum_{i,k=1}^{n} \frac{\partial}{\partial w_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_j^i}\right)\mathrm{d}v_j^i \wedge \mathrm{d}w_{j+l}^k\, h \\
&\quad - \sum_{j,l=-\infty}^{\infty}\sum_{i,k=1}^{n} \frac{\partial}{\partial v_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_j^i}\right)\mathrm{d}v_j^i \wedge \mathrm{d}v_{j+l}^k\, h
\end{aligned}$$

and using

$$\frac{\partial}{\partial w_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_j^i}\right) = \frac{\partial}{\partial w_j^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_{j+l}^i}\right) \qquad \frac{\partial}{\partial v_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_j^i}\right) = \frac{\partial}{\partial v_j^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_{j+l}^i}\right)$$

$$\frac{\partial}{\partial v_{j+l}^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta w_j^i}\right) = \frac{\partial}{\partial w_j^k}\left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta v_{j+l}^i}\right)$$

we obtain the conservation of the symplectic form.

This is a generalization of the symplectic structure for the Hamiltonian ODEs [11] to the infinite set of semidiscrete equations. Let us note that recently proposed multi-symplectic formulation of PDEs [12–15] allows one to consider local conservation of symplecticity, which we do not have in the present framework.

Underlying equations (1.1) possess the conservation of symplecticity

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} = 0 \qquad \boldsymbol{\omega} = \int \mathrm{d}\boldsymbol{v} \wedge \mathrm{d}\boldsymbol{w}\,\mathrm{d}x = \int \sum_{i=1}^{n} \mathrm{d}v^i \wedge \mathrm{d}w^i\,\mathrm{d}x \tag{3.18}$$

that is the continuous limit of (3.17). In this case $\mathrm{d}\boldsymbol{v}$ and $\mathrm{d}\boldsymbol{w}$ are solutions of the variational equations for (1.1)

$$\mathrm{d}v_t^i = \hat{\mathrm{d}}\left(\frac{\delta\mathcal{H}}{\delta w^i}\right) = \sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial w_l^k}\left(\frac{\delta\mathcal{H}}{\delta w^i}\right)\mathrm{d}w_l^k + \sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial v_l^k}\left(\frac{\delta\mathcal{H}}{\delta w^i}\right)\mathrm{d}v_l^k$$

$$\mathrm{d}w_t^i = -\hat{\mathrm{d}}\left(\frac{\delta\mathcal{H}}{\delta v^i}\right) = -\sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial w_l^k}\left(\frac{\delta\mathcal{H}}{\delta v^i}\right)\mathrm{d}w_l^k - \sum_{k=1}^{n}\sum_{l} \frac{\partial}{\partial v_l^k}\left(\frac{\delta\mathcal{H}}{\delta v^i}\right)\mathrm{d}v_l^k$$

where the operator $\hat{\mathrm{d}}$ denotes the vertical differential of the vertical form it acts on. In our case we have vertical 0-forms, i.e. functions. We assume that $\mathrm{d}\boldsymbol{v}$, $\mathrm{d}\boldsymbol{w}$ and their space derivatives which appear in the variational equations decay at $x \to \infty$. Let us show that the conservation of symplecticity using the variational complex (see, for example, [2]):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} &= \int \left(\sum_{i=1}^{n} \mathrm{d}v_t^i \wedge \mathrm{d}w^i + \sum_{i=1}^{n} \mathrm{d}v^i \wedge \mathrm{d}w_t^i\right)\mathrm{d}x \\ &= \int \left(\sum_{i=1}^{n} \hat{\mathrm{d}}\left(\frac{\delta\mathcal{H}}{\delta w^i}\right) \wedge \mathrm{d}w^i - \sum_{i=1}^{n} \mathrm{d}v^i \wedge \hat{\mathrm{d}}\left(\frac{\delta\mathcal{H}}{\delta v^i}\right)\right)\mathrm{d}x \\ &= \int \hat{\mathrm{d}}\left(\sum_{i=1}^{n} \frac{\delta\mathcal{H}}{\delta v^i}\,\mathrm{d}v^i + \frac{\delta\mathcal{H}}{\delta w^i}\,\mathrm{d}w^i\right)\mathrm{d}x = \delta \int \left(\sum_{i=1}^{n} \frac{\delta\mathcal{H}}{\delta v^i}\,\mathrm{d}v^i + \frac{\delta\mathcal{H}}{\delta w^i}\,\mathrm{d}w^i\right)\mathrm{d}x.\end{aligned}$$

Using integration by parts, we obtain that

$$\int \sum_{i=1}^{n} \frac{\delta\mathcal{H}}{\delta v^i}\,\mathrm{d}v^i\,\mathrm{d}x = \int \sum_{i=1}^{n}\sum_{k} (-D_x)^k \frac{\partial H}{\partial v_k^i}\,\mathrm{d}v^i\,\mathrm{d}x = \int \sum_{i=1}^{n}\sum_{k} \frac{\partial H}{\partial v_k^i}\,\mathrm{d}v_k^i\,\mathrm{d}x$$

because $\mathrm{d}v_k^i$, $i = 1, \ldots, n$ tend to zero as $x \to \infty$. Since the same result is valid for variables $\boldsymbol{w}$ we obtain

$$\int \left(\sum_{i=1}^{n} \frac{\delta\mathcal{H}}{\delta v^i}\,\mathrm{d}v^i + \frac{\delta\mathcal{H}}{\delta w^i}\,\mathrm{d}w^i\right)\mathrm{d}x = \int \hat{\mathrm{d}}H\,\mathrm{d}x = \delta\mathcal{H}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} = \delta^2\mathcal{H} = 0$$

as follows from the exactness of the variational complex.

3.3.2. Conservation of distinguished functionals.

Definition 3.4. *For a given Hamiltonian operator* $\underset{h}{\mathcal{D}}$ *a distinguished functional is a functional* $\underset{h}{\mathcal{C}}(x, h, \underset{h}{\boldsymbol{u}}^{(n)})$ *such that*

$$\underset{h}{\mathcal{D}}\left(\frac{\delta \underset{h}{\mathcal{C}}}{\delta \boldsymbol{u}}\right) = 0. \tag{3.19}$$

It follows that a functional is distinguished if and only if its Poisson bracket with every other functional is trivial:

$$\{\underset{h}{\mathcal{C}}, \underset{h}{\mathcal{H}}\}_h = 0 \qquad \text{for any} \quad \underset{h}{\mathcal{H}} \in \underset{h}{\mathcal{F}}. \tag{3.20}$$

Distinguished functionals are conserved by semidiscrete equations originating from any Hamiltonian functional. For the canonical bracket there are no non-trivial distinguished functionals so that we do not obtain conservation laws of this type.

3.3.3. Hamiltonian form of Noether's theorem.

Definition 3.5. *The Hamiltonian vector field associated with a functional* $\underset{h}{\mathcal{P}}$ *is the unique smooth vector field* X_P *satisfying*

$$X_P(\underset{h}{\mathcal{F}}) = \{\underset{h}{\mathcal{F}}, \underset{h}{\mathcal{P}}\}_h. \tag{3.21}$$

In the coordinate form it can be presented as the operator

$$X_P = \underset{h}{\mathcal{D}}\left(\frac{\delta \underset{h}{\mathcal{P}}}{\delta u^i}\right)\frac{\partial}{\partial u^i}. \tag{3.22}$$

Some symmetries (3.2) are given as Hamiltonian vector fields or are equivalent to Hamiltonian vector fields under factorization (3.16). Such vector fields let us use the following theorem [3].

Theorem 3.6. *For a Hamiltonian system of semidiscrete evolution equations (2.12) a Hamiltonian vector field* X_P *determines a generalized symmetry of the system if and only if there is an equivalent functional* $\underset{h}{\tilde{\mathcal{P}}} = \underset{h}{\mathcal{P}} - \underset{h}{\mathcal{C}}$, *differing from* $\underset{h}{\mathcal{P}}$ *only by a time-dependent distinguished functional* $\underset{h}{\mathcal{C}}(t, x, h, u^{(n)})$, *such that* $\underset{h}{\tilde{\mathcal{P}}}$ *determines a conservation law.*

For the operator J generating the canonical bracket a Hamiltonian vector field has the form

$$X_P = \frac{\delta \underset{h}{\mathcal{P}}}{\delta w^i}\frac{\partial}{\partial v^i} - \frac{\delta \underset{h}{\mathcal{P}}}{\delta v^i}\frac{\partial}{\partial w^i} \tag{3.23}$$

where $\underset{h}{\mathcal{P}}$ is the generating functional. The canonical bracket has only trivial time-dependent functionals, i.e. functions $f(t)$. Thus, for a Hamiltonian symmetry (3.23) there corresponds a conservation law $\underset{h}{\tilde{\mathcal{P}}} = \underset{h}{\mathcal{P}} - f(t)$, where the function $f(t)$ needs to be found with the help of the considered equations. In a particular case when the Hamiltonian functional and the considered Hamiltonian symmetry are time independent we obtain a linear function $f(t) = at + b$, $a, b = \text{constant}$.

4. The nonlinear wave equation

In this section we consider the nonlinear wave equation

$$v_{tt} = v_{xx} - V'(v) \tag{4.1}$$

where $V(v)$ is some smooth function. For simplicity, we consider the case of scalar v. With the help of a new variable $w = v_t$, equation (4.1) can be rewritten as the system

$$\begin{aligned} v_t &= w \\ w_t &= v_{xx} - V'(v). \end{aligned} \tag{4.2}$$

This is a canonical Hamiltonian system generated by the Hamiltonian functional

$$\mathcal{H} = \int \left(\frac{w^2}{2} + \frac{v_x^2}{2} + V(v) \right) \mathrm{d}x. \tag{4.3}$$

Let us take the following approximation of the Hamiltonian functional:

$$\underset{h}{\mathcal{H}} = \sum_{\Omega} H[v, w]h \qquad H = \frac{w^2}{2} + \frac{\underset{h}{v}_1^{\,2}}{2} + V(v). \tag{4.4}$$

It provides us with a system of semidiscrete equations,

$$\begin{aligned} \dot{v} &= w \\ \dot{w} &= \underset{h}{v}_2 - V'(v). \end{aligned} \tag{4.5}$$

For arbitrary $V(v)$ the admitted transformation group for (4.5) is two dimensional. Its Lie algebra is spanned by the operators

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x}. \tag{4.6}$$

The Lorentz transformation

$$X_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$$

admitted by the underlying system (4.2) with arbitrary $V(v)$ [16], is lost under discretization since it breaks the mesh invariance. Thus in the case of arbitrary $V(v)$ Noether's theorem gives two conservation laws:

(a) The time translation X_1 leads to the conservation of the Hamiltonian functional $\underset{h}{\mathcal{H}}$. Indeed, the factorization of operator X_1 gives the evolutionary vector field

$$\bar{X}_1 = w\frac{\partial}{\partial v} + \left(\underset{h}{v}_2 - V'(v)\right)\frac{\partial}{\partial w}$$

which is generated by the Hamiltonian functional. Thus we obtain conservation of the Hamiltonian functional (4.4) that denotes the conservation of energy.

(b) The space translation X_2 corresponds to the evolutionary operator

$$\bar{X}_2 = \underset{h}{\tilde{D}}_0(v)\frac{\partial}{\partial v} + \underset{h}{\tilde{D}}_0(w)\frac{\partial}{\partial w}. \tag{4.7}$$

Checking condition (3.23) for the coefficients of operator $\bar{X}_2$, we find the generating functional

$$\underset{h}{\mathcal{P}}_2 = \sum_{\Omega} w \underset{h}{\tilde{D}}_0(v)h = -\sum_{\Omega} v \underset{h}{\tilde{D}}_0(w)h. \qquad (4.8)$$

This functional is a conservation law of the semidiscrete equations (4.5). In the continuous limit it corresponds to the functional

$$\mathcal{P}_2 = \int w v_x \,\mathrm{d}x = -\int v w_x \,\mathrm{d}x \qquad (4.9)$$

which is a conservation law of equations (4.2). In physical applications this conservation law is referred to as linear momentum.

In order to obtain this conservation law we need to consider 'non-local' functionals, i.e. functionals which are defined as double sums over the mesh points, since operator $\underset{h}{\tilde{D}}_0$ is a sum over infinitely many mesh points.

Let us consider the special cases of the potential $V(v)$ which lead to additional Hamiltonian symmetries

(a) For the quadratic potential $V(v) = Cv^2/2$ there is an infinite series of conservation laws for the wave system (4.5). In this case the semidiscrete system admits the infinite set of Hamiltonian operators

$$Y_k = \underset{h}{D}_0(\underset{h}{v}_{2k})\frac{\partial}{\partial v} + \underset{h}{D}_0(\underset{h}{w}_{2k})\frac{\partial}{\partial w} \qquad k = 0, 1, \ldots$$

which provides us with the infinite set of conserved functionals

$$\underset{h}{R}_k = \sum_{\Omega} w \underset{h}{D}_0(\underset{h}{v}_{2k})h = -\sum_{\Omega} v \underset{h}{D}_0(\underset{h}{w}_{2k})h.$$

In the continuous limit these conservation laws correspond to the functionals

$$R_k = \int v w_{(2k+1)} \,\mathrm{d}x$$

which are conserved quantities for the system (4.2).

Since for the quadratic potential we obtain the system of linear equations their solutions possess a superposition principle. It is reflected in the invariance with respect to the symmetry

$$Z = \alpha(t,x)\frac{\partial}{\partial v} + \alpha_t(t,x)\frac{\partial}{\partial w}$$

where the function $\alpha(t,x)$ is an arbitrary solution of the equation

$$\alpha_{tt}(t,x) = \frac{\alpha(t,x+h) - 2\alpha(t,x) + \alpha(t,x-h)}{h^2} - C\alpha(t,x). \qquad (4.10)$$

The operator Z is Hamiltonian. It corresponds to the functional

$$\underset{h}{T} = \sum_{\Omega}(\alpha(t,x)w - \alpha_t(t,x)v)h$$

which is a conservation law of the semidiscrete linear system if function $\alpha(t,x)$ satisfies equation (4.10). In the continuous limit this functional goes to the functional

$$T = \int (\alpha(t,x)w - \alpha_t(t,x)v)\,\mathrm{d}x$$

which is a conservation law of (4.2) if the function $\alpha(t,x)$ satisfies the equation

$$\alpha_{tt}(t,x) = \alpha_{xx}(t,x) - C\alpha(t,x). \qquad (4.11)$$

Table 1. A number of conservation laws for the wave system (4.5) without a potential $V'(v) \equiv 0$, which correspond to particular cases of the symmetry $Z = \alpha(t, x)\frac{\partial}{\partial v} + \alpha_t(t, x)\frac{\partial}{\partial w}$.

Function $\alpha(t, x)$	Operator Z	Discrete conservation law	Continuous conservation law
1	$\frac{\partial}{\partial v}$	$\sum_\Omega wh$	$\int w\,\mathrm{d}x$
t	$t\frac{\partial}{\partial v} + \frac{\partial}{\partial w}$	$\sum_\Omega (tw - v)h$	$\int (tw - v)\,\mathrm{d}x$
x	$x\frac{\partial}{\partial v}$	$\sum_\Omega xwh$	$\int xw\,\mathrm{d}x$
tx	$tx\frac{\partial}{\partial v} + x\frac{\partial}{\partial w}$	$\sum_\Omega (txw - xv)h$	$\int (txw - xv)\,\mathrm{d}x$
$t^2 + x^2$	$(t^2 + x^2)\frac{\partial}{\partial v} + 2t\frac{\partial}{\partial w}$	$\sum_\Omega ((t^2 + x^2)w - 2tv)h$	$\int ((t^2 + x^2)w - 2tv)\,\mathrm{d}x$

(b) Let us consider the case $C = 0$, i.e. the wave system (4.5) without a potential $V'(v) \equiv 0$, in detail. The symmetry Z is specified by a solution of the equation

$$\alpha_{tt}(t, x) = \frac{\alpha(t, x + h) - 2\alpha(t, x) + \alpha(t, x - h)}{h^2}. \tag{4.12}$$

We present a number of conservation laws corresponding to the symmetry Z taking particular solutions $\alpha(t, x)$ of equation (4.12) in table 1.

Let us note that the continuous conservation laws in the case $V'(v) \equiv 0$ are given by the function $\alpha(t, x)$ satisfying equation (4.11) with $C = 0$. The general solution can be written down as

$$\alpha(t, x) = \alpha_1(t - x) + \alpha_2(t + x)$$

where α_1 and α_2 are arbitrary functions.

Example 4.1. As we mentioned before the factorized operator may fail to be admitted. Let us consider $V'(v) \equiv 0$. In this case system (4.5) admits the scaling symmetry

$$X_* = -t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + w\frac{\partial}{\partial w}.$$

The corresponding evolutionary vector field

$$\bar{X}_* = (tw + x\underset{h}{\tilde{D}}{}_0(v))\frac{\partial}{\partial v} + (w + t\underset{h}{v}{}_2 + x\underset{h}{\tilde{D}}{}_0(w))\frac{\partial}{\partial w}$$

found with the help of the factorization formula (3.16), is Hamiltonian. It is generated by the functional

$$\underset{h}{\mathcal{P}}{}_* = t\,\underset{h}{\mathcal{H}} + \sum_\Omega xw\underset{h}{\tilde{D}}{}_0(v)h.$$

However, neither the functional $\underset{h}{\mathcal{P}}{}_*$ is a conservation law nor the canonical operator $\bar{X}_*$ is a symmetry of the semidiscrete system (4.5) with $V'(v) \equiv 0$. It happens because the operator

$$x * D^0 = x\frac{\partial}{\partial x} + x\underset{h}{\tilde{D}}{}_0(v)\frac{\partial}{\partial v} + x\underset{h}{\tilde{D}}{}_0(w)\frac{\partial}{\partial w} + \cdots$$

which is used for the factorization

$$\bar{X}_* = X_* - t * D_t - x * D^0$$

is not admitted by the semidiscrete system: the second equation does not allow this operator since

$$2\underset{h}{v}_2 + x\underset{h}{\tilde{D}}_0(\dot{w}) \neq \underset{-h}{D}\,\underset{+h}{D}(x\underset{h}{\tilde{D}}_0(v)) \qquad (4.13)$$

on the solutions of the semidiscrete equations. In the limit $h \to 0$ equation (4.13) turns into an equality and the continuous limit of $\underset{h}{\mathcal{P}}_*$, namely the functional

$$\mathcal{P}_* = t\,\mathcal{H} + \int x w v_x \,\mathrm{d}x$$

is a conservation law of the system (4.2) with $V' \equiv 0$. ◊

5. The nonlinear Schrödinger equation

Another equation which can be cast into the canonical Hamiltonian form is the nonlinear Schrödinger equation (1.3) which arises in nonlinear optics. It describes the main features of the beam interaction with a nonlinear medium and is considered as the *basic equation of nonlinear optics* [17]. The equation also has important applications in plasma physics [18].

Let us consider the case of one-dimensional space,

$$\mathrm{i}\psi_t + \psi_{xx} + F'(|\psi|^2)\psi = 0. \qquad (5.1)$$

For real and imaginary components v and w ($\psi = v + \mathrm{i}w$) the Schrödinger equation can be rewritten as the system

$$\begin{aligned} v_t &= -w_{xx} - F'(v^2 + w^2)w \\ w_t &= v_{xx} + F'(v^2 + w^2)v \end{aligned} \qquad (5.2)$$

which is a canonical Hamiltonian system. It is generated by the Hamiltonian functional

$$\mathcal{H} = \tfrac{1}{2}\int \left(|\psi_x|^2 - F(|\psi|^2)\right)\mathrm{d}x. \qquad (5.3)$$

The system of equations (5.2) in the case of arbitrary F admits a four-dimensional transformation group presented by the operators [17]:

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x} \qquad X_3 = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w} \qquad X_4 = 2t\frac{\partial}{\partial x} - xw\frac{\partial}{\partial v} + xv\frac{\partial}{\partial w}. \qquad (5.4)$$

In the case $F(z) = Cz^2/2$ there is an additional scaling symmetry

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}. \qquad (5.5)$$

Physically, this particular case of $F(z)$ corresponds to an isotropic medium with the cubic polarizability (first approximation of the nonlinear polarizability).

Let us discretize the Hamiltonian functional as

$$\underset{h}{\mathcal{H}} = \sum_{\Omega} H[v, w]h \qquad H = \tfrac{1}{2}\left(\underset{h}{v}_1^2 + \underset{h}{w}_1^2 - F(v^2 + w^2)\right) \qquad (5.6)$$

then we obtain the system of semidiscrete equations

$$\begin{aligned} \dot{v} &= -\underset{h}{w}_{2} - F'(v^2 + w^2)w \\ \dot{w} &= \underset{h}{v}_{2} + F'(v^2 + w^2)v. \end{aligned} \tag{5.7}$$

Now we are in a position to go through the symmetries (5.4) and (5.5), to check whether they are preserved under the space discretization. If they are admitted by the semidiscrete system (5.7), they may provide conservation laws according to theorem 3.6.

(a) The time translation X_1 is admitted by system (5.7). The evolutionary symmetry corresponding to X_1 is generated by the functional (5.6). Thus we find that the Hamiltonian functional is a conservation law of equations (5.7). Its continuous limit is the functional $\mathcal{H}$. Physically, we interpret this conservation law as conservation of energy.

(b) The space translation X_2 corresponds to the evolutionary operator (4.7), which we have already examined. This symmetry leads us to the conservation law $\underset{h}{\mathcal{P}}_2$ (see (4.8)).

(c) The evolutionary symmetry X_3 is generated by the functional

$$\underset{h}{\mathcal{P}}_3 = \frac{1}{2} \sum_{\Omega} \left(v^2 + w^2\right) h \tag{5.8}$$

which is conservation of mass for the semidiscrete system (5.7). In the continuous limit it goes to the functional

$$\mathcal{P}_3 = \frac{1}{2} \int \left(v^2 + w^2\right) \mathrm{d}x. \tag{5.9}$$

(d) The Galilean transformation X_4 is not admitted in the semidiscrete case. Its coefficients violate the second mesh invariance condition in (3.4). The action of the generated by X_4 transformation on the independent variables

$$\hat{t} = t \qquad \hat{x} = x + 2ta$$

clearly shows that it destroys the mesh geometry. It follows that the continuous conservation law corresponding to the movement of the centre of mass

$$\mathcal{P}_4 = \int \left(\frac{1}{2}x(v^2 + w^2) + t(wv_x - vw_x)\right) \mathrm{d}x \tag{5.10}$$

has no counterpart in the semidiscrete case.

(e) The additional symmetry X_5 is admitted by equations (5.7) with quadratic F, but it is not Hamiltonian (both in the continuous and discrete cases).

The considered symmetries and their generating functionals are shown in table 2.

6. Conclusions

In the paper we have considered semidiscrete canonical Hamiltonian equations and shown how to find conservation laws of such equations using Noether's theorem. Our interest in canonical Hamiltonian equations is also motivated by their connection with Euler–Lagrange equations [1].

Many PDEs can be presented as Euler–Lagrange equations for appropriate Lagrangian functionals [2, 19] and can be rewritten as canonical Hamiltonian equations (1.1). Let us

Table 2. Some symmetries and their generating functionals for the canonical bracket. The functionals (up to some time-dependent functions) are conservation laws of the semidiscrete canonical Hamiltonian equations.

Operator	Operator in evolutionary form	Functional	Continuous limit of functional
$\frac{\partial}{\partial t}$	$\frac{\delta \underset{h}{\mathcal{H}}}{\delta w}\frac{\partial}{\partial v} - \frac{\delta \underset{h}{\mathcal{H}}}{\delta v}\frac{\partial}{\partial w}$	$\underset{h}{\mathcal{H}}$	$\mathcal{H}$
$\frac{\partial}{\partial x}$	$\underset{h}{\tilde{D}}_0(v)\frac{\partial}{\partial v} + \underset{h}{\tilde{D}}_0(w)\frac{\partial}{\partial w}$	$\sum_\Omega w\underset{h}{\tilde{D}}_0(v)h$	$\int wv_x\,\mathrm{d}x$
	$w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}$	$\frac{1}{2}\sum_\Omega(v^2+w^2)h$	$\frac{1}{2}\int(v^2+w^2)\,\mathrm{d}x$
	$\underset{h}{D}_0(\underset{h}{v}_{2k})\frac{\partial}{\partial v} + \underset{h}{D}_0(\underset{h}{w}_{2k})\frac{\partial}{\partial w}$	$\sum_\Omega w\underset{h}{D}_0(\underset{h}{v}_{2k})h$	$\int wv_{(2k+1)}\,\mathrm{d}x$
	$\alpha(t,x)\frac{\partial}{\partial v} + \alpha_t(t,x)\frac{\partial}{\partial w}$	$\sum_\Omega(\alpha(t,x)w - \alpha_t(t,x)v)h$	$\int(\alpha(t,x)w - \alpha_t(t,x)v)\,\mathrm{d}x$

illustrate this on the example of the nonlinear wave equation (4.1) which is the Euler–Lagrange equation

$$\frac{\delta L}{\delta v} = 0 \qquad \frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} \tag{6.1}$$

for the Lagrangian functional

$$\mathcal{L} = \iint L(v, v_t, v_x)\,\mathrm{d}x\,\mathrm{d}t \qquad L = \frac{v_t^2}{2} - \frac{v_x^2}{2} - V(v). \tag{6.2}$$

With the help of the Legendre transformation

$$w = \frac{\partial L}{\partial v_t} = v_t \qquad \mathcal{H} = \int\left(\frac{\partial L}{\partial v_t}v_t - L\right)\mathrm{d}x$$

we obtain the Hamiltonian functional (4.3). It generates equations (4.2), which are equivalent to (4.1).

A similar connection between Euler–Lagrange equations and canonical Hamiltonian equations can be established in the semidiscrete case. Let us consider the semidiscretization of functional (6.2)

$$\underset{h}{\mathcal{L}} = \int\sum_\Omega \underset{h}{L}(v, \dot{v}, \underset{h}{v}_1)h\,\mathrm{d}t \qquad \underset{h}{L} = \frac{\dot{v}^2}{2} - \frac{\underset{h}{v}_1^{\,2}}{2} - V(v).$$

Its Euler–Lagrange equation

$$\frac{\delta \underset{h}{\mathcal{L}}}{\delta v} = 0 \qquad \frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial \dot{v}} - \underset{-h}{D}\frac{\partial}{\partial \underset{h}{v}_1} \tag{6.3}$$

has the form

$$\ddot{v} = \underset{h}{v}_2 - V'(v). \tag{6.4}$$

We can introduce the discrete Hamiltonian functional

$$w = \frac{\partial \underset{h}{L}}{\partial \dot{v}} = \dot{v} \qquad \underset{h}{\mathcal{H}} = \sum_\Omega\left(\frac{\partial \underset{h}{L}}{\partial \dot{v}}\dot{v} - \underset{h}{L}\right)h$$

which in our case is the functional (4.4). It generates the semidiscrete evolution equations (4.5), which are equivalent to (6.4).

Thus, looking for conservation laws of the semidiscrete Euler–Lagrange equations, one can consider the equivalent canonical Hamiltonian systems and find conservation laws in the Hamiltonian framework. They have the form

$$\int \mathcal{P}\,\mathrm{d}x \qquad \text{and} \qquad \sum_{\Omega} \underset{h}{\mathcal{P}}\, h \tag{6.5}$$

in the continuous and semidiscrete cases, respectively. If necessary, the conservation laws can be rewritten in terms of the original variables. For example, the conservation laws of the semidiscrete wave system (4.5) with an arbitrary potential $V(v)$ found in section 4 can be rewritten in terms of variables used in the Lagrangian approach

$$\underset{h}{\mathcal{P}}_1 = \sum_{\Omega}\left(\frac{\dot{v}^2}{2} + \frac{\underset{h}{v}_1^2}{2} + V(v)\right)h \qquad \underset{h}{\mathcal{P}}_2 = \sum_{\Omega} \dot{v}\,\underset{h}{\tilde{D}}_0(v)h.$$

For a quadratic potential we also have the infinite series of conservation laws

$$\underset{h}{R}_k = \sum_{\Omega} \dot{v}\,\underset{h}{D}_0(\underset{h}{v}_{2k})h = -\sum_{\Omega} v\,\underset{h}{D}_0(\underset{h}{\dot{v}}_{2k})h \qquad k = 0, 1, 2, \ldots$$

and the conservation laws

$$\underset{h}{T} = \sum_{\Omega}(\alpha(t,x)\dot{v} - \alpha_t(t,x)v)h$$

where the function $\alpha(t,x)$ must satisfy equation (4.10).

One of the advantages of the Lagrangian approach over the Hamiltonian one is the possibility of finding local conservation. In the Hamiltonian framework we can find only global conservation laws of the form (6.5). This advantage of the Lagrangian approach is difficult to represent for the semidiscrete equations because as we have seen some conservation laws have densities which involve discrete presentations of the continuous derivatives and, consequently, are not local.

Here we considered symplecticity for the semidiscrete canonical Hamiltonian equations. It is interesting to examine whether one can introduce an analogous structure for semidiscrete Lagrangian equations and study its connection with the symplectic form similar to [20], where such a connection was established for discrete mechanical systems on Lie groups. It would also be interesting to investigate the connection between semidiscrete Hamiltonian equations and semidiscrete equations with soliton solutions [21–23] as well as features of integrable equations known for other (not semidiscrete) equations [24, 25].

For simplicity we considered only the case of one-dimensional space. The extension to the multi-dimensional space is straightforward.

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